

UNITED STATES DEPARTMENT OF THE INTERIOR BUREAU OF MINES HELIUM ACTIVITY HELIUM RESEARCH CENTER INTERNAL REPORT

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 $Z_r = (Z_o/P_o) f N_{P=0}^r P_r$

BY

В.	J. Dalton	n
R	E. Barie	au

BRANCH

Fundamental Research Branch

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HELIUM RESEARCH CENTER

INTERNAL REPORT

DERIVATION OF FORMULAS FOR EVALUATING THE STANDARD ERRORS IN B, C, AND $N_{P=0}$ WHICH APPEAR IN THE EQUATION:

$$Z_r = (Z_o/P_o) f N_{P=0}^r P_r$$

By

B. J. Dalton and Robert E. Barieau

Branch of Fundamental Research

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by

B. J. Dalton and Robert E. Barieau $\frac{2}{}$

ABSTRACT

The method of treating a set of isothermally measured pressures P_o , P_1 , P_2 , ..., P_r , for a Burnett experiment, consists of expressing the compressibility factor isotherm of the gas in terms of a function of either P or ρ and evaluating the volume ratio, $N_{P=0}$, and the constants in the function by least squares solution. This report gives the method of least squares as applied to the fundamental equation for the Burnett experiment, taking into consideration the change in $N_{P=0}$ with pressure.

Formulas are given for evaluating the standard deviation of a single $\mathbf{P}_{\mathbf{r}}$, the standard deviation of each of the constants evaluated, and the standard deviation of the compressibility factor.

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Work on manuscript completed September 1965.

DERIVATION OF FORMULAS FOR EVALUATING THE STANDARD ERRORS IN B, C, AND $N_{p=0}$ WHICH APPEAR IN THE EQUATION: $Z_r = (Z_o/r_o) i N_{p=0}^r P_r$

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E. J. Dalton and Robert E. Barteau 2/

ABSTRACT

The method of treating a set of isothermally measured prossures P_0 , P_1 , P_2 , ..., P_r , for a Burnett experiment, consists of expressing the compressibility factor isotherm of the gas in terms of a function of either P or ρ and evaluating the volume ratio, $N_{P=0}$, and the constants in the function by least squares solution. This report gives the method of least squares as applied to the fundamental equation for the Burnett experiment, taking into consideration the change in $N_{P=0}$ with pressure.

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Work on manuscript completed September 1365.

INTRODUCTION

The method of treating Burnett type data consists of expressing the compressibility factor isotherm of the gas in terms of a function of P or ρ and evaluating the best values for the constants in the function by least squares solution.

In a previous report $(\underline{6})^{3/}$, we have given formulas for eval-

3/ Underlined numbers in parentheses refer to items in the list of references at the end of this report.

uating the best values for the constants appearing in the fundamental equation for the Burnett experiment: $Z_r = (Z_0/P_0)N^r P_r$, assuming N, the cell constant, to be independent of the pressure.

Canfield (5, 7) has pointed out that the volume ratio varies with pressure due to distortion of the bombs and a shift in the null point of the differential pressure indicator. This change in the cell constant, for a given expansion, was expressed by the equation

$$N_{r} = N_{P=0} \left(\frac{1 + \alpha P_{r}}{1 + \beta P_{r-1}} \right)$$
 (1)

where

N_{P=0} = volume ratio at zero pressure

 $N_r = \text{volume ratio for the } \frac{\text{th}}{\text{expansion}}$

 $P_r = pressure after the r + th expansion$

 P_{r-1} = pressure before the $r = \frac{th}{r}$ expansion

The method of treating Surnett type data consists of expressing the compressibility factor isotherm of the gas in terms of a nunction of P or p and evaluating the best values for the constants in the function by least equares splution.

In a previous report (a) 3/, we have given formulas for eval-

3/ Underliped numbers in parenthoses refer to Itoms in the List of references at the end of this report:

usting the best values for the constants appearing in the fundamental equation for the Surmett experiments $Z_{\mu} = (Z_{\mu}/P_{\mu})N^{T}P_{\mu}$, assuming N, the dell constant, to be independent of the pressure.

Canfield (5, 2) has pointed out that the volume ratio varies with pressure due to distortion of the bombs and a shift in the null point of the differential pressure indicator. This change in the cell constant, for a given expansion, was expressed by the equation

(1)
$$\left(\frac{1-3}{3}+1\right)^{1-3}$$
 $\left(\frac{1}{3}+1\right)^{1-3}$ $\left(\frac{1}{3}+1\right)^{1-3}$

sasme.

N_{pro} = volume ratio at zero pressure

N_r = volume ratio for the rth expansion

P_r = pressure after the rth expansion

$$\alpha P_r$$
 = change in the total volume,
 $(V_1 + V_2)_{P=P_r}$ divided by the
total volume at zero pressure
 βP_{r-1} = change in V_1 at $P=P_{r-1}$ divided
 V_1 at $P=0$

Therefore, representing the change in the volume ratio by equation

(1), the fundamental equation for the Burnett experiment assumes

the form

$$z_{r} = (z_{o}/P_{o})N_{1} \cdot N_{2} \cdot \dots N_{r}P_{r}$$
 (2)

Since

$$N_1 = N_{P=0} \left(\frac{1 + \alpha P_1}{1 + \beta P_0} \right)$$

$$N_2 = N_{P=0} \left(\frac{1 + \alpha P_2}{1 + \beta P_1} \right)$$

2 = (2/Pg) + Ppo 2

 $N_{r} = N_{P=0} \left(\frac{1 + \alpha P_{r}}{1 + \beta P_{r-1}} \right)$

equation (2) is expressible as

$$Z_{r} = (Z_{o}/P_{o}) f N_{P=0}^{r} P_{r}$$
 (3)

where

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Therefore, representation to the charge in the volume ratio by equation (1), the limit experience, assumes the form

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$$f = \frac{(1 + \alpha P_{1(obs)})(1 + \alpha P_{2(obs)}) \dots (1 + \alpha P_{r(obs)})}{(1 + \beta P_{o(obs)})(1 + \beta P_{1(obs)}) \dots (1 + \beta P_{r-1(obs)})}$$
(4)

Associated with this function, equation (3), whose constants have been evaluated by least squares, the following errors are of interest: the standard deviation of a single $P_{r(obs)}$; the standard deviation of each of the constants; and the standard deviation of compressibility factors.

The method outlined in reference 1 served as the basis for evaluating the least squares solution for the constants appearing in equation (3). The methods outlined in reference 2 were used for evaluating the above-mentioned errors.

METHOD OF OBTAINING THE LEAST SQUARES SOLUTION FOR THE CONSTANTS APPEARING IN THE EQUATION: $Z_r = (Z_o/P_o)fN_{P=0}^r$

The fundamental equation for the Burnett experiment, assuming the variation of the volume ratio with pressure to be expressible by equation (1), is of the form

$$Z_{r} = (Z_{o}/P_{o}) f N_{P=0}^{r} P_{r}$$
(3)

where f is defined by equation (4).

Now there are two series expansion which can be employed for representing the compressibility factor isotherm: the Leiden expansion in powers of ρ and the Berlin expansion in powers of P. The Berlin expansion

$$Z_{r} = 1 + BP_{r} + CP_{r}^{2} + \dots$$
 (5)

(3) (may 1-20 + 1) ... (may 120 + 1) (may 120 + 2) - 2

Associated with this fameless, equation (3), whose constants have been swalteness by least squares, the following orders are of interact, the attention of a single P₁(che); the stanton of sevietima of the congressibility factors.

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MEMOR OF CHICAGO IN THE LINES SQUARES SOLUTION TO GOMESIA CONSTRUCT A PRESENTED IN THE CONSTRUCT S. = (2/2/2) | 7/2/2

The fundamiental equation for the luminous experiment, assuming
the variation of the volume ratio will pressure to be expressible
by equation (1), is of the form

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there f to derigon by equation (A).

Now there are two sories expansion which can be employed for representing the compressibility factor incluence the Labelle Labelle appareton in powers of p. The Seriis expansion in powers of P. The Seriis expansion

Z = 1 + np + op; + op; + . . .

(2)

$$Z_o = 1 + BP_o + CP_o^2 + \dots$$
 (6)

was chosen because all of the parameters for which we seek a least squares solution can be expressed in terms of the original observations. It was assumed that the series expansion in powers of P could be truncated after the third virial coefficient.

Suppose we let the functional relationship between the variables, P_r and r, involving the three parameters, $N_{p=0}$, B, C, be

$$F = F(r, P_r, N_{p=0}, B, C) = 0$$
 (7)

Now because of random errors in the observed pressures, when $P_{r(obs)}$ is substituted in the above expression, F will not be exactly zero. Let F_r be the value of F when the observed values of r and P_r are substituted in equation (7). Thus,

$$F_r = F(r, P_{r(obs)}, N_{p=0}, B, C)$$
 (8)

Now we assume that r, the expansion number, is accurately known. Therefore, equation (7) may be solved for $P_{r(calc)}$ so that equation (7) is exactly satisfied. Thus,

$$F = F(r, P_{r(calc)}, N_{p=0}, B, C) \equiv 0$$
 (9)

and $P_{r(calc)}$ is the solution of equation (9).

Now ΔP_r , the residual of P_r , is the difference between the observed and calculated values. This is not the true random error in our observed P_r because we do not know the true value of P_r . However, we can maximize the probability that the ΔP_r 's are equal

value of the second of the representation of the contained and the contained observances. It was not the chart the second of the property of the second of t

Suppose we has the forestends relationship between the waterlander, E. and a land or land of the page of the C. be

Now because of candem series in the observed pressures, when Peteles, in entire to exactly series in the the value of 1 when the observed values of r and P, and substituted in aquation (2). Thus,

How we minute that to the asymmetric mades, is accurately known.

Therefore, equation (3) may be solved for P_e(male) so that some

$$T = T(t_1, T_{t_1, t_2, t_3}) \cdot T_{t_1, t_2} \cdot T_{t_1, t_2} = 0$$
 (9)

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to the true random errors, and this is just what the principle of least squares does. The principle of least squares says that we maximize the probability that the ΔP_r 's represent the true random errors by minimizing the sum of the squares of the weighted residuals. Thus, we should minimize the function

$$R = \sum_{r=1}^{r} W_{P_{r}(obs)} (\Delta P_{r})^{2}$$
 (10)

and evaluate $N_{P=0}$, B, and C so that

$$\left(\frac{\partial R}{\partial B}\right)_{r,P_{r(obs)},C,N_{p=0}} = 2\sum_{r=1}^{r} W_{P_{r(obs)}} \Delta P_{r}\left(\frac{\partial \Delta P_{r}}{\partial B}\right) = 0 \quad (11)$$

$$\left(\frac{\partial R}{\partial C}\right)_{r,P_{r(obs)},B,N_{p=0}} = 2\sum_{r=1}^{r} W_{P_{r(obs)}} \Delta P_{r}\left(\frac{\partial \Delta P_{r}}{\partial C}\right) = 0 \quad (12)$$

$$\left(\frac{\partial R}{\partial N_{P=0}}\right)_{r,P_{r(obs)},B,C} = 2\sum_{r=1}^{r} W_{P_{r(obs)}} \Delta P_{r} \left(\frac{\partial \Delta P_{r}}{\partial N_{P=0}}\right) = 0$$
 (13)

In equation (10), $W_{P_r(obs)}$ is the weight of be assigned to r(obs) the observed P_r . If the P_r 's all have the same precision index, then they will have the same weight and $W_{P_r(obs)}$ = 1. If the P_r 's do not all have the same precision index, then

$$W_{\text{Pr(obs)}} = \frac{L^2}{S_{\text{Pr(obs)}}^2}$$

where L is a constant and $S_{r(obs)}^2$ is the variance of $P_{r(obs)}$.

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(21)
$$0 = \left(\frac{2^{2}}{2^{2}}\right)_{1} 2^{2}$$
 (23) $0 = \left(\frac{2^{2}}{2^{2}}\right)_{1} 2^{2}$ (24) $0 = \left(\frac{2^{2}}{2^{2}}\right)_{1} 2^{2}$ (25)

In equation (10), We will have the same precision trains of the Park and the same precision trains, and then other other same weight and described the Park and described the same security to the sam

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In a particular problem, it may be necessary to assume $W_{P_{r(obs)}} = 1$ in the beginning. However, if this is done the residuals, $Y_{i} = [P_{r(obs)} - P_{r(calc)}]$, should be examined to see if there is any statistical evidence for the residuals squared being a function of $P_{r(obs)}$. Any assumption as to the variance being a function of $P_{r(obs)}$ can always be checked by examining the residuals. In any event, $W_{P_{r(obs)}}$ is not a function of the constants to be evaluated.

In order to evaluate $N_{P=0}$, B, C, we need to linearize Y_i with respect to the undetermined constants. A truncated Taylor's series expansion was used to do this.

In a previous report (1), we show that the linearized normal equations are expressible as

$$a_1 \Delta B + b_1 \Delta C + c_1 \Delta N_{P=0} = m_1$$
 (14)

$$a_2 \triangle B + b_2 \triangle C + c_2 \triangle N_{P=0} = m_2$$
 (15)

$$a_3 \Delta B + b_3 \Delta C + c_3 \Delta N_{P=0} = m_3$$
 (16)

Equations (14), (15), and (16) result from expanding Y_i , $(\partial Y_i/\partial B)$, $(\partial Y_i/\partial C)$, and $(\partial Y_i/\partial N_{P=0})$ about an approximate solution Y_i^0 , ignoring second and higher order derivatives. The linearized coefficients ΔB , ΔC , $\Delta N_{P=0}$ are defined as

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$$\Delta B = B - B^{O}$$

$$\Delta C = C - C^{O}$$

$$\Delta N_{P=0} = N_{P=0} - N_{P=0}^{O}$$
(17)

where B, C, $N_{P=0}$ are the undetermined constants and B°, C°, $N_{P=0}^{o}$ are approximate values for these quantities.

The a's, b's, c's, and m's appearing in the normal equations are given as (1):

$$a_{1} = \sum_{r=1}^{r} W_{P_{r(obs)}} \left[\left(\frac{\partial Y_{i}}{\partial B} \right)^{o^{2}} + Y_{i}^{o} \left(\frac{\partial^{2} Y_{i}}{\partial B^{2}} \right)^{o} \right]$$
 (18)

$$a_{2} = b_{1} = \sum_{r=1}^{r} W_{P_{r(obs)}} \left[\left(\frac{\partial Y_{i}}{\partial B} \right)^{o} \left(\frac{\partial Y_{i}}{\partial C} \right)^{o} + Y_{i}^{o} \left(\frac{\partial^{2} Y_{i}}{\partial B \partial C} \right)^{o} \right]$$
(19)

$$a_{3} = c_{1} = \sum_{r=1}^{r} W_{P_{r(obs)}} \left[\left(\frac{\partial Y_{i}}{\partial B} \right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}} \right)^{o} + Y_{i}^{o} \left(\frac{\partial^{2} Y_{i}}{\partial B \partial N_{P=0}} \right)^{o} \right]$$
(20)

$$b_{2} = \sum_{r=1}^{r} W_{P_{r}(obs)} \left[\left(\frac{\partial Y_{i}}{\partial C} \right)^{o^{2}} + Y_{i}^{o} \left(\frac{\partial^{2} Y_{i}}{\partial C^{2}} \right)^{o} \right]$$
 (21)

$$b_{3} = c_{2} = \sum_{r=1}^{r} W_{P_{r}(obs)} \left[\left(\frac{\partial Y_{i}}{\partial C} \right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}} \right)^{o} + Y_{i}^{o} \left(\frac{\partial^{2} Y_{i}}{\partial C \partial N_{P=0}} \right)^{o} \right]$$
(22)

$$c_{3} = \sum_{r=1}^{r} W_{P_{r}(obs)} \left[\left(\frac{\partial Y_{i}}{\partial N_{P=0}} \right)^{o^{2}} + Y_{i}^{o} \left(\frac{\partial^{2} Y_{i}}{\partial N_{P=0}^{2}} \right)^{o} \right]$$
 (23)

where B. C. Ng-p are the undetermined opasteats and B', C', Fp-p are approximate values for three quantities.

The s's, b's, c's, and m's uppearing in the normal equations we often as (1):

$$m_{1} = -\sum_{r=1}^{r} W_{P_{r}(obs)} Y_{i}^{o} \left(\frac{\partial Y_{i}}{\partial B}\right)^{o}$$
 (24)

$$m_2 = -\sum_{r=1}^{r} W_{P_{r(obs)}} Y_i^o \left(\frac{\partial Y_i}{\partial C}\right)^o$$
 (25)

$$m_3 = -\sum_{r=1}^{r} W_{P_r(obs)} Y_i^o \left(\frac{\partial Y_i}{\partial N_{P=0}}\right)^o$$
 (26)

where

$$\left(\frac{\partial Y_{i}}{\partial B}\right)^{O} = \frac{\left(\partial F/\partial B\right)^{O}}{\left(\partial F/\partial P_{r(calc)}\right)^{O}}$$
(27)

$$\left(\frac{\partial Y_{i}}{\partial C}\right)^{O} = \frac{\left(\partial F/\partial C\right)^{O}}{\left(\partial F/\partial P_{r(calc)}\right)^{O}}$$
(28)

$$\left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{O} = \frac{\left(\partial F/\partial N_{P=0}\right)^{O}}{\left(\partial F/\partial P_{r(calc)}\right)^{O}}$$
(29)

$$\left(\frac{\partial^{2} \mathbf{r}}{\partial \mathbf{B}^{2}}\right)^{\circ} = \begin{bmatrix} \frac{\partial^{2} \mathbf{r}}{\partial \mathbf{B}^{2}} & \frac{\partial^{2} \mathbf{r}}{\partial \mathbf{$$

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$$\left(\frac{\partial^{2} Y_{i}}{\partial C^{2}}\right)^{\circ} = \frac{\left(\partial^{2} F/\partial C^{2}\right)^{\circ} - \frac{2(\partial^{2} F/\partial C\partial P_{r(calc)})^{\circ}(\partial F/\partial C)^{\circ}}{\left[(\partial F/\partial P_{r(calc)})^{\circ}\right]^{2}} + \frac{(\partial F/\partial C)^{\circ} (\partial^{2} F/\partial P_{r(calc)})^{\circ}}{\left[(\partial F/\partial P_{r(calc)})^{\circ}\right]^{3}}$$
(31)

$$\frac{\left(\frac{\partial^{2} F}{\partial N_{p=0} \partial B}\right)^{\circ}}{\left(\frac{\partial F}{\partial P_{r}(calc)}\right)^{\circ}} - \frac{\left(\frac{\partial^{2} F}{\partial B \partial P_{r}(calc)}\right)^{\circ} \left(\frac{\partial F}{\partial N_{p=0}}\right)^{\circ}}{\left[\left(\frac{\partial F}{\partial P_{r}(calc)}\right)^{\circ}\right]^{2}}$$

$$\left(\frac{\partial^{2} Y_{i}}{\partial N_{p=0} \partial B}\right)^{\circ} = \left(\frac{\partial^{2} Y_{i}}{\partial B \partial N_{p=0}}\right)^{\circ} = -\frac{\left(\frac{\partial F}{\partial B}\right)^{\circ} \left(\frac{\partial^{2} F}{\partial P_{r}(calc)}\right)^{\circ} \left(\frac{\partial N_{p=0}}{\partial P_{r}(calc)}\right)^{\circ}}{\left[\left(\frac{\partial F}{\partial P_{r}(calc)}\right)^{\circ}\right]^{2}}$$

$$+ \frac{\left(\frac{\partial F}{\partial B}\right)^{\circ} \left(\frac{\partial F}{\partial N_{p=0}}\right)^{\circ} \left(\frac{\partial^{2} F}{\partial P_{r}(calc)}\right)^{\circ}}{\left[\left(\frac{\partial F}{\partial P_{r}(calc)}\right)^{\circ}\right]^{3}}$$
(32)

$$\left(\frac{\partial^{2} F/\partial N_{P=0} \partial C)^{\circ}}{(\partial F/\partial P_{r(calc)})^{\circ}} - \frac{(\partial^{2} F/\partial C \partial P_{r(calc)})^{\circ} (\partial F/\partial N_{P=0})^{\circ}}{(\partial F/\partial P_{r(calc)})^{\circ}}^{2}} \right)$$

$$\left(\frac{\partial^{2} Y_{i}}{\partial N_{P=0} \partial C}\right)^{\circ} = \left(\frac{\partial^{2} Y_{i}}{\partial C \partial N_{P=0}}\right)^{\circ} = -\frac{(\partial F/\partial C)^{\circ} (\partial^{2} F/\partial P_{r(calc)})^{\partial N_{P=0}})^{\circ}}{\left[(\partial F/\partial P_{r(calc)})^{\circ}\right]^{2}} + \frac{(\partial F/\partial C)^{\circ} (\partial F/\partial N_{P=0})^{\circ} (\partial^{2} F/\partial P_{r(calc)}^{2})^{\circ}}{\left[(\partial F/\partial P_{r(calc)})^{\circ}\right]^{3}}$$

$$\left(\frac{\partial F/\partial C}{\partial F/\partial N_{P=0}}\right)^{\circ} (\partial^{2} F/\partial P_{r(calc)}^{2})^{\circ}$$

$$\left(\frac{\partial^{2} Y_{i}}{\partial N_{P=0}^{2}}\right)^{\circ} = \frac{2(\partial^{2} F/\partial N_{P=0}^{2})^{\circ} - 2(\partial^{2} F/\partial N_{P=0$$

$$\left(\frac{\partial^{2} \mathbf{r}}{\partial \mathbf{r}}\right)^{\circ} = \left(\frac{\partial^{2} \mathbf{r}}{\partial \mathbf{B} \partial \mathbf{C}}\right)^{\circ} = \frac{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{B} \partial \mathbf{C}}\right)^{\circ} - \frac{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ} - \frac{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ}}{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ}} - \frac{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ} - \frac{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ}}{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ}} - \frac{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ} - \frac{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ}}{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ}} - \frac{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ}}{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ}} - \frac{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ}}{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ}} - \frac{\left(\partial^{2} \mathbf{r}}{\partial \mathbf{C} \partial \mathbf{C}\right)^{\circ}}{\left(\partial^{2} \mathbf$$

$$Y_{i}^{o} = [P_{r(obs)} - P_{r(calc)}]^{o}$$
(36)

Now in order to evaluate the solutions of our linearized normal equations, we need values of first and second derivatives of the function, F,

$$F = F(r, P_{r(calc)}, N_{P=0}, B, C) \equiv 0$$
 (9)

and

$$\frac{(3^{2}F/3N_{P=0}^{2})^{\circ}}{(3F/3E_{r}(calc))^{\circ}} = \frac{2(3^{2}F/3N_{P=0}^{2})^{\circ}(3F/3N_{P=0}^{2})^{\circ}}{((3F/3E_{r}(calc))^{\circ})^{\circ}}$$

$$\frac{(3F/3E_{r}(calc))^{\circ}}{(3F/3E_{r}(calc))^{\circ}} = \frac{(3F/3E_{r}(calc))^{\circ}}{((3F/3E_{r}(calc))^{\circ})^{\circ}}$$

$$(34)$$

$$\frac{(3^{2}r/3c)^{9}}{(3F/3P_{r}(calc))^{9}} = \frac{(3^{2}r/3c)^{9}}{(3F/3P_{r}(calc))^{9}} = \frac{(3^{2}r/3c)^{9}}{(3F/3P_{r}(calc))^{9}} = \frac{(3^{2}r/3c)^{9}}{(3F/3C)^{9}} = \frac{(3^{2}r/3c)^{9}}{(3F/3C)^{9}} = \frac{(3^{2}r/3c)^{9}}{(3F/3C)^{9}} = \frac{(3^{2}r/3c)^{9}}{(3F/3C)^{9}} = \frac{(3^{2}r/3c)^{9}}{(3F/3C)^{9}} = \frac{(3^{2}r/3c)^{9}}{(3F/3P_{r}(calc))^{9}} = \frac{(3^{2}r/3c)^{9}}{(3F/3P_$$

$$y_1^0 = [P_{r(obs)} - P_{r(cate)}]^0$$
 (36)

New in order to evaluate the solutions of our linearized normal equations, we need values of first and second derivatives of the function. F.

$$r = r(r, r_{colc}), r_{r=0}, s, c) = 0$$
 (9)

$$F = Z_{r(calc)} - (Z_{o}/P_{o})f_{(calc)} N_{P=0}^{r} P_{r(calc)} = 0$$

$$= 1 + BP_{r(calc)} + CP_{r(calc)}^{2} - \left(\frac{1 + BP_{o} + CP_{o}^{2}}{P_{o}}\right)f_{(calc)}N_{P=0}^{r} P_{r(calc)}$$
(37)

where f (calc) of equation (37) is given as

$$f_{\text{(calc)}} = \frac{(1 + \alpha P_{1(\text{calc})})(1 + \alpha P_{2(\text{calc})}) \dots (1 + \alpha P_{r(\text{calc})})}{(1 + \beta P_{0})(1 + \beta P_{1(\text{calc})}) \dots (1 + \beta P_{r-1_{(\text{calc})}})}$$
(38)

Therefore, from equations (37) and (38), the first and second derivatives of F are:

$$\left(\frac{\partial F}{\partial B}\right)_{r,C,N_{p=0},P_{r(calc)}} = P_{r(calc)}(1 - f_{(calc)}N_{p=0}^{r})$$

$$\left(\frac{\partial F}{\partial C}\right)_{r,B,N_{p=0},P_{r(calc)}} = P_{r(calc)} \left[P_{r(calc)} - f_{(calc)}N_{p=0}^{r}P_{o}\right]$$

$$\left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r(calc)}} = -r(Z_{o}/P_{o})f_{(calc)}^{r-1}P_{p=0}^{r-1}P_{r(calc)}$$

$$\left(\frac{\partial F}{\partial P_{r(calc)}}\right)_{r,B,C,N_{p=0}} = B + 2CP_{r(calc)} - \left(\frac{Z_{o}}{P_{o}}\right)f_{(calc)}N_{p=0}^{r}\left[1 + \frac{\alpha^{p}_{r(calc)}}{(1 + \alpha^{p}_{r(calc)})}\right]$$

$$\left(\frac{\partial^2 F}{\partial B^2}\right)_{r,C,N_{p=0},P_{r(calc)}} = 0$$

where Eccepts of equations (37) is given as

(80) (olas)s (-10 - 10 - 10 (olas)) (0 + 1) (olas) (0 + 1) (olas)

Therefore, for equations (37) and (38), the first and second

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$$\left(\frac{\partial^2 F}{\partial c^2}\right)_{r,B,N_{p=0},P_{r(calc)}} = 0$$

$$\left(\frac{\partial^2 F}{\partial N_{P=0}^2}\right)_{r,B,C,P_{r(calc)}} = -r(r-1)\left(\frac{Z_o}{P_o}\right) f_{(calc)} N_{P=0}^{r-2} P_{r(calc)}$$

$$\left(\frac{\partial^2 F}{\partial P_{r(calc)}^2}\right)_{r,B,C,N_{p=0}} = 2C - 2\left(\frac{Z_o}{P_o}\right) f_{(calc)} N_{p=0}^r \left(\frac{\alpha}{1 + \alpha P_{r(calc)}}\right)$$

$$\left[\frac{\partial}{\partial C} \left(\frac{\partial F}{\partial B}\right)_{r,C,N_{P=0},P_{r(calc)}}\right]_{r,B,N_{P=0},P_{r(calc)}} = 0$$

$$\left[\frac{\partial}{\partial N_{P=0}} \left(\frac{\partial F}{\partial B}\right)_{r,C,N_{P=0},P_{r(calc)}}\right]_{r,B,C,P_{r(calc)}} = -r f_{(calc)}^{r-1} P_{r(calc)}$$

$$\left[\frac{\partial}{\partial N_{P=0}} \left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r(calc)}}\right]_{r,B,C,P_{r(calc)}} = -r P_{o} f_{(calc)}^{N_{P=0}^{r-1}} P_{r(calc)}$$

$$\left[\frac{\partial}{\partial P_{r(calc)}} \left(\frac{\partial F}{\partial B}\right)_{r,C,N_{p=0},P_{r(calc)}}\right]_{r,B,C,N_{p=0}} = 1 - f_{(calc)} N_{p=0}^{r} \left[1 + \frac{\alpha P_{r(calc)}}{1 + \alpha P_{r(calc)}}\right]$$

$$\left[\frac{\partial}{\partial P_{r(calc)}}\left(\frac{\partial F}{\partial C}\right)_{r,B,N_{p=0},P_{r(calc)}}\right]_{r,B,C,N_{p=0}} = 2P_{r(calc)} - f_{(calc)}N_{p=0}^{r}P_{p=0}P_{o}\left[1 + \frac{\alpha P_{r(calc)}}{1 + \alpha P_{r(calc)}}\right]_{r,B,C,N_{p=0}}$$

$$\left[\frac{\partial}{\partial P_{r(calc)}} \left(\frac{\partial F}{\partial N_{p=0}}\right)_{r,B,C,P_{r(calc)}}\right]_{r,B,C,N_{p=0}} = -r\left(\frac{z_{o}}{P_{o}}\right)_{f(calc)} + \frac{\alpha^{p}_{r(calc)}}{1 + \alpha^{p}_{r(calc)}}$$

$$\left[\frac{\partial}{\partial B}\left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r(calc)}}\right]_{r,C,N_{P=0},P_{r(calc)}} = \left[\frac{\partial}{\partial C}\left(\frac{\partial F}{\partial B}\right)_{r,C,N_{P=0},P_{r(calc)}}\right]_{r,B,N_{P=0},P_{r(calc)}}$$

$$\left[\frac{\partial}{\partial B}\left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r(calc)}}\right]_{r,C,N_{P=0},P_{r(calc)}} = \left[\frac{\partial}{\partial N_{P=0}}\left(\frac{\partial F}{\partial B}\right)_{r,C,N_{P=0},P_{r(calc)}}\right]_{r,B,C,P_{r(calc)}}$$

$$\left[\frac{\partial}{\partial C} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r(calc)}}\right]_{r,B,N_{P=0},P_{r(calc)}} = \left[\frac{\partial}{\partial N_{P=0}} \left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r(calc)}}\right]_{r,B,C,P_{r(calc)}}$$

$$\left[\frac{\partial}{\partial N_{P=0}} \left(\frac{\partial F}{\partial P_{r(calc)}}\right)_{r,B,C,N_{P=0}}\right]_{r,B,C,P_{r(calc)}} = \left[\frac{\partial}{\partial P_{r(calc)}} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r(calc)}}\right]_{r,B,C,N_{P=0}}$$

$$\frac{\partial}{\partial B} \left(\frac{\partial F}{\partial P_{r(calc)}} \right)_{r,B,C,N_{p=0}} = \left[\frac{\partial}{\partial P_{r(calc)}} \left(\frac{\partial F}{\partial B} \right)_{r,C,N_{p=0},P_{r(calc)}} \right]_{r,B,C,N_{p=0}}$$

Therefore, equations (27) - (35) can be expressed in terms of the original observations by the following expressions:

$$\left(\frac{\partial Y_{i}}{\partial B}\right)^{o} = \frac{P_{r(calc)}\left[1 - f_{(calc)}N_{P=0}^{r}\right]}{B + 2CP_{r(calc)} - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)}N_{P=0}^{r}\left[1 + \frac{\alpha P_{r(calc)}}{(1 + \alpha P_{r(calc)})}\right]}$$

Therefore, equations (27) - (35) can be expressed in Leros of

$$\frac{\left(\frac{2^{N}}{68R}\right)^{2} - \left(\frac{2^{N}}{68R}\right)^{2} - \left(\frac{2^{N}}{68R}\right)^{2}}{8 + 2C^{2}\epsilon(calc)} - \left(\frac{2^{N}}{6}\right)^{2}(calc)^{N}_{peo} \left(\frac{1}{4} + \frac{2^{N}}{68R}\right)^{2} + \frac{2C^{2}}{68R}\left(\frac{2^{N}}{68R}\right)^{2} + \frac{2C^{2}}{68R}\left(\frac{$$

$$\left(\frac{\partial Y_{i}}{\partial C}\right)^{o} = \frac{P_{r(calc)} \left[P_{r(calc)} - f_{(calc)}N_{P=0}^{r} P_{o}\right]}{B + 2CP_{r(calc)} - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)}N_{P=0}^{r} \left[1 + \frac{\alpha P_{r(calc)}}{(1 + \alpha P_{r(calc)})}\right]}$$

$$\left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{O} = \frac{-r(Z_{o}/P_{o}) f_{(calc)} N_{P=0}^{r-1} P_{r(calc)}}{B + 2CP_{r(calc)} - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)} N_{P=0}^{r} \left[1 + \frac{\alpha P_{r(calc)}}{(1 + \alpha P_{r(calc)})}\right]$$

$$\frac{-2P_{r(calc)}(1-f_{(calc)}N_{P=0}^{r})\left[1-f_{(calc)}N_{P=0}^{r}\left(1+\frac{\alpha^{P}_{r(calc)}}{(1+\alpha^{P}_{r(calc)})}\right)\right]}{B+2CP_{r(calc)}-\left(\frac{Z_{o}}{P_{o}}\right)f_{(calc)}N_{P=0}^{r}\left[1+\frac{\alpha^{P}_{r(calc)}}{(1+\alpha^{P}_{r(calc)})}\right]^{2}}$$

$$\left(\frac{\partial^2 Y_i}{\partial B^2}\right)^{o} =$$

$$+ 2P_{r(calc)}^{2} \left(1 - f_{(calc)}N_{P=0}^{r}\right)^{2} \left[C - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)}N_{P=0}^{r} \frac{\alpha}{(1 + \alpha P_{r(calc)})}\right]$$

$$B + 2CP_{r(calc)} - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)}N_{P=0}^{r} \left[1 + \frac{\alpha P_{r(calc)}}{(1 + \alpha P_{r(calc)})}\right]^{3}$$

$$\frac{P_{r(ca1c)}}{(\frac{3c}{3c})^{2}} = \frac{P_{r(ca1c)} - F_{(ca1c)}^{R_{po}} - F_{ca1c)}^{R_{po}}}{(\frac{3c}{3c})^{2}} = \frac{P_{r(ca1c)}^{R_{po}} - P_{r(ca1c)}^{R_{po}}}{(\frac{3c}{3c})^{2}} = \frac{P_{r(ca1c)}^{R_{po}} - P_{r(ca1c)}^{R_{po}}}{(\frac{3c}{3c})^{2}}} = \frac{P_{r(ca1c)}^{R_{po}} - P_{r(ca1c)}^{R_{po}}}{(\frac{3c}{3c})^{2}} = \frac{P_{r(ca1c)}^{R_{po}}}{(\frac{3c}{3c})^{2}} = \frac{P_{r(ca1c)}^{R_{po}}}{(\frac{3c}{3c})^{2}}$$

$$\frac{dV_{1}}{dN_{p=0}} = \frac{-r(2_{o}/r_{o})^{\frac{p}{2}}(calc)^{\frac{p}{2}-0} \frac{r_{calc}}{r_{calc}}}{1 + 2cP_{r(calc)} - (\frac{2_{o}}{r_{o}})^{\frac{p}{2}}(calc)^{\frac{p}{2}-0} \frac{r_{calc}}{r_{calc}}} = \frac{V_{r(calc)}}{1 + cP_{r(calc)}} = \frac{V_{r($$

$$+ 2p_{r(calc)}^{2} (1 - \epsilon_{(calc)}^{2} n_{p=0}^{2})^{2} (c - (\frac{\epsilon_{0}}{R})^{2} \epsilon_{(calc)}^{2} n_{p=0}^{2} (\frac{1 + \epsilon_{0}}{\epsilon_{r(calc)}^{2}})^{2}$$

$$+ 2p_{r(calc)}^{2} - (\frac{\epsilon_{0}}{R})^{2} \epsilon_{(calc)}^{2} n_{p=0}^{2} (\frac{1 + \epsilon_{0}}{R} \epsilon_{calc})^{2}$$

$$\frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o} \left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}}{\left[\frac{\partial^{2} Y_{i}}{\partial c^{2}} \right]^{o}} = \frac{\partial^{2} Y_{i}}{\partial c^{2}} = \frac{\partial^{2} Y_{i}}{\partial c^{2}} = \frac{\partial^{2} Y_{i}}{\partial c^{2}} = \frac{\partial^{2}$$

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$$\frac{\left(\frac{\partial^{2}Y_{i}}{\partial N_{P=0}^{2}}\right)^{\circ} f_{(calc)} f_$$

$$\frac{1}{2} \left[\frac{1}{2} + 2C_{1}^{2} \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\frac{1}{2} \right) + \left(\cos 1c \right) + \left(\cos 1$$

$$\frac{-P_{r(calc)}(P_{r(calc)} - f_{(calc)} N_{P=0}^{r} P_{o}) \left[1 - f_{(calc)} N_{P=0}^{r} \left(1 + \frac{\alpha P_{r(calc)}}{(1 + \alpha P_{r(calc)})}\right)\right]}{\left[B + 2CP_{r(calc)} - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)} N_{P=0}^{r} \left(1 + \frac{\alpha P_{r(calc)}}{(1 + \alpha P_{r(calc)})}\right)\right]}$$

$$\frac{-P_{r(calc)}(1 - f_{(calc)} N_{P=0}^{r}) \left[2P_{r(calc)} - f_{(calc)} N_{P=0}^{r} P_{o} \left(1 + \frac{\alpha P_{r(calc)}}{(1 + \alpha P_{r(calc)})}\right)\right]}{\left[B + 2CP_{r(calc)} - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)} N_{P=0}^{r} \left(1 + \frac{\alpha P_{r(calc)}}{(1 + \alpha P_{r(calc)})}\right)\right]^{2}}$$

$$\frac{+2P_{r(calc)}^{2}(1 - f_{(calc)} N_{P=0}^{r}) \left(P_{r(calc)} - f_{(calc)} N_{P=0}^{r} P_{o}\right) \left[C - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)} N_{P=0}^{r} \frac{\alpha}{(1 + \alpha P_{r(calc)})}\right]}{\left[B + 2CP_{r(calc)} - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)} N_{P=0}^{r} P_{o}\right) \left[C - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)} N_{P=0}^{r} \frac{\alpha}{(1 + \alpha P_{r(calc)})}\right]}$$

$$\left(\frac{\partial^2 Y_i}{\partial C \partial B}\right)^{o} = \left(\frac{\partial^2 Y_i}{\partial B \partial C}\right)^{o} =$$

(Sans

$$\left(\frac{\sigma^{2} Y_{i}}{\partial N_{P=0} \partial B}\right)^{\circ} = \left(\frac{\sigma^{2} Y_{i}}{\partial B \partial N_{P=0}}\right)^{\circ} = \frac{\left(\frac{\sigma^{2} Y_{i}}{P_{o}}\right) f_{(calc)} N_{P=0}^{r-1} P_{r(calc)} \left(1 + \frac{\sigma^{2} P_{r(calc)}}{(1 + \sigma^{2} P_{r(calc)})}\right)}{\left[B + 2CP_{r(calc)} N_{P=0}^{r-1} P_{r(calc)} \left[1 - f_{(calc)} N_{P=0}^{r} \left(1 + \frac{\sigma^{2} P_{r(calc)}}{(1 + \sigma^{2} P_{r(calc)})}\right)\right]} \right]$$

$$\left[B + 2CP_{r(calc)} N_{P=0}^{r-1} P_{r(calc)} \left[1 - f_{(calc)} N_{P=0}^{r} \left(1 + \frac{\sigma^{2} P_{r(calc)}}{(1 + \sigma^{2} P_{r(calc)})}\right)\right]^{2} \right]$$

$$\left[B + 2CP_{r(calc)} N_{P=0}^{r-1} N_{P=0}^{r-1} N_{P=0}^{r-1} \left(1 + \frac{\sigma^{2} P_{r(calc)}}{(1 + \sigma^{2} P_{r(calc)})}\right)\right]^{2} \right]$$

$$\left[B + 2CP_{r(calc)} - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)} N_{P=0}^{r-1} \left(1 + \frac{\sigma^{2} P_{r(calc)}}{(1 + \sigma^{2} P_{r(calc)})}\right)\right]^{2} \right]$$

$$\left[B + 2CP_{r(calc)} N_{P=0}^{r-1} N_{P=0}^{r-1} N_{P=0}^{r-1} \left[C - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)} N_{P=0}^{r-1} N_{P=0}^{r-1} \left[C - \left(\frac{Z_{o}}{P_{o}}\right) f_{(calc)} N_{P=0}^{r-1} N$$

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$$\begin{bmatrix} -r & P_{o} & f_{(calc)} & N_{P=0}^{r-1} & P_{r(calc)} \\ [B + 2CP_{r(calc)} - (\frac{Z_{o}}{P_{o}})f_{(calc)} & N_{P=0}^{r} & (1 + \frac{\alpha^{P}_{r(calc)}}{(1 + \alpha^{P}_{r(calc)})}) \end{bmatrix} \\ + r(\frac{Z_{o}}{P_{o}})f_{(calc)}N_{P=0}^{r-1} & P_{r(calc)} & [2P_{r(calc)} - f_{(calc)}N_{P=0}^{r} & P_{o} & (1 + \frac{\alpha^{P}_{r(calc)}}{(1 + \alpha^{P}_{r(calc)})})] \\ + r(\frac{Z_{o}}{P_{o}})f_{(calc)}N_{P=0}^{r-1} & P_{r(calc)} & [2P_{r(calc)} - f_{(calc)}N_{P=0}^{r} & P_{o} & (1 + \frac{\alpha^{P}_{r(calc)}}{(1 + \alpha^{P}_{r(calc)})})] \\ + r(\frac{Z_{o}}{P_{o}})f_{(calc)}N_{P=0}^{r-1} & P_{o} & (1 + \frac{\alpha^{P}_{r(calc)}}{(1 + \alpha^{P}_{r(calc)})}) \end{bmatrix}^{2} \\ + P_{r(calc)}(P_{r(calc)} - f_{(calc)}N_{P=0}^{r} & P_{o}) & r(\frac{Z_{o}}{P_{o}})f_{(calc)}N_{P=0}^{r-1} & (1 + \frac{\alpha^{P}_{r(calc)}}{(1 + \alpha^{P}_{r(calc)})}) \end{bmatrix}^{2} \\ - 2P_{r(calc)}^{2}(P_{r(calc)} - f_{(calc)}N_{P=0}^{r} & P_{o}) & r(\frac{Z_{o}}{P_{o}})f_{(calc)}N_{P=0}^{r-1} & [C - (\frac{Z_{o}}{P_{o}})f_{(calc)}N_{P=0}^{r} & \frac{\alpha}{(1 + \alpha^{P}_{r(calc)})}) \end{bmatrix}^{2} \\ - P_{r(calc)}(P_{r(calc)} - f_{(calc)}N_{P=0}^{r} & P_{o}) & r(\frac{Z_{o}}{P_{o}})f_{(calc)}N_{P=0}^{r-1} & [C - (\frac{Z_{o}}{P_{o}})f_{(calc)}N_{P=0}^{r} & \frac{\alpha}{(1 + \alpha^{P}_{r(calc)})}) \end{bmatrix}^{2}$$

SATE OF SCORE

Now in evaluating the best values for B, C, and $N_{P=0}$, we solve the linearized normal equations by an iterative procedure. Even though it is necessary to solve equations (14), (15), and (16) by a series of approximations, it is important to realize that the best values for the constants are determined such that the normal equations are exactly satisfied.

The solutions to equations (14), (15), and (16) are (1):

$$D_{o} \Delta B = D_{1}^{m_{1}} + D_{2}^{m_{2}} + D_{3}^{m_{3}}$$
 (39)

$$D_{o} \triangle C = D_{4}^{m} + D_{5}^{m} + D_{6}^{m}$$
 (40)

$$D_{o}^{\Delta N}_{P=0} = D_{7}^{m}_{1} + D_{8}^{m}_{2} + D_{9}^{m}_{3}$$
 (41)

where

$$D_1 = b_2 c_3 - b_3 c_2 \tag{42}$$

$$D_4 = D_2 = b_3 c_1 - b_1 c_3 \tag{43}$$

$$D_7 = D_3 = b_1 c_2 - b_2 c_1 \tag{44}$$

$$D_5 = a_1 c_3 - a_3 c_1 \tag{45}$$

$$D_8 = D_6 = a_2 c_1 - a_1 c_2 \tag{46}$$

$$D_9 = a_1 b_2 - a_2 b_1 \tag{47}$$

$$D_{o} = D_{1}a_{1} + D_{2}a_{2} + D_{3}a_{3}$$
 (48)

Now in evaluating the best values for S. C. and N_{p=Q}, we note the linearized normal equations by an inerative procedure. Evan though it is nuccessary to solve equations (14), (15), and (16) by a soriet of approximations, it is important to realize that the normal boost values for the constants are determined such that the normal equations are exactly entiated.

The volutions to equations (14), (15), and (16) are (1):

$$D_{\mu}\Delta C = D_{\mu}m_{\mu} + D_{\mu}m_{\mu} + D_{\mu}m_{\mu}$$
 (40)

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$$D_1 = b_1 c_3 - b_3 c_2$$
 (42)

$$D_{1} = D_{2} = b_{3}c_{1} - b_{1}c_{3}$$
 (43)

EXPRESSIONS FOR DETERMINING VARIANCES AND COVARIANCES OF THE CONSTANTS EVALUATED

We now proceed to evaluate the variances and covariances of the constants evaluated. To do this, we apply the law for the "Propagation of Errors" (3, 8). This law states that if we have a function or quantity, say Q, that is a function of the independently observed quantities y_1, y_2, \ldots , then the variance of the quantity Q is given as

$$s_{Q}^{2} = \sum_{i=1}^{n} \left(\frac{\partial Q}{\partial y_{i(obs)}}\right)^{2} s_{y_{i(obs)}}^{2}$$
(49)

where S_Q^2 is the variance of Q and $S_{i(obs)}^2$ is the variance of $y_{i(obs)}$. Extracting the square root of the variance, we obtain a value on the same scale as the original measurements. This value, S_Q , is called the standard error or the standard deviation of Q.

The value of the constant B, which we have evaluated is a function of all of the observed r's and of all of the observed P_r 's. Since we have assumed that r is accurately known, then the expression for the variance in B is given by the equation

$$s_{B}^{2} = \sum_{r=1}^{r} \left(\frac{\partial B}{\partial P_{r(obs)}}\right)^{2} s_{P_{r(obs)}}^{2}$$
 (50)

and there will be an equation similar to equation (50) for determining the variance of C and the variance of $N_{P=0}$.

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We now proceed to evaluate the variances and covariances of the constants evaluated. To do this, we apply the law for the "Propagation of Serors" (g. g). This law states that if we have a tunction or quantity, may 0, that is a function of the independently observed quantities y, sy, ..., then the variance of the quantities of the given as

where So is the variance of Q and So is the variance of Yi(obs)

Yi(obs) Extracting the square root of the variance, we obtain a value on the same scale as the original measurements. This value, So, is called the standard error or the standard deviation of Q.

The value of the constant B, which we have evaluated is a function of all of the observed P's. Since we have assumed that r is accurately known, than the expression for the variance in B is given by the equation

$$S_{B}^{2} = \sum_{r=1}^{2} \left(\frac{m}{\delta P_{r}(obs)} \right)^{2} S_{P}^{2}(obs)$$
(50)

and there will be an equation similar to equation (50) for determining the variance of C and the variance of N In order to evaluate equation (50), we must evaluate $(\partial B/\partial P_{r(obs)})$ for each $P_{r(obs)}$, multiply this quantity by $S_{r(obs)}$, square the product, and then sum the product over all of the observed P_r 's.

In a previous report (2), we have outlined the details for evaluating the variances and all of the covariances of the constants evaluated.

For our particular problem, these variances and covariances are determined from the following relations (2):

$$S_{B}^{2} = \frac{L^{2}}{D_{o}^{2}} + D_{3}^{2} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{o} + D_{2}^{2} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \right)^{2} + 2D_{1}D_{2} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{o} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} + 2D_{1}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{o} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}D_{3} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{2}$$

$$S_{C}^{2} = \frac{L^{2}}{D_{o}^{2}} + D_{6}^{2} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{\circ} + D_{5}^{2} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \right)^{2} + 2D_{4}D_{5} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} + 2D_{4}D_{5} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} + 2D_{4}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} + 2D_{5}D_{6} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{\circ} \right)^{\circ}$$

In order to evaluate equation (50), we must evaluate (50/2) (50/2) (50/2) for each Process and this quantity by Reaquare the product, and then sem the product over all of the observed Processing

In a provious report (2), we have cutilized the details for evaluating the variances and all of the covertances of the constants evaluated.

For our particular problem, these variances and covertences are determined from the following relations (2):

$$\frac{\left[\frac{1}{2} \frac{1}{2$$

$$\frac{1}{2} \frac{1}{2} \frac{1$$

$$S_{N_{P=0}}^{2} = \frac{L^{2}}{D_{o}^{2}} + D_{g}^{2} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{o} + D_{g}^{2} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \right)^{2} + 2D_{7}D_{8} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{o} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} + 2D_{7}D_{8} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{o} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} + 2D_{7}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{o} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}\right)^{o} + 2D_{8}D_{9} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial N_{P=0}}$$

$$S_{BC}^{2} = \frac{L^{2}}{D_{o}^{2}} + (D_{2}D_{4} + D_{1}D_{5}) \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial R}\right)^{2} + D_{2}D_{5} \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{2} + (D_{2}D_{4} + D_{1}D_{5}) \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial R}\right)^{2} \left(\frac{\partial Y_{i}}{\partial B}\right)^{2} + (D_{1}D_{6} + D_{3}D_{4}) \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{2} \left(\frac{\partial Y_{i}}{\partial C}\right)^{2} + (D_{2}D_{6} + D_{3}D_{5}) \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{2} \left(\frac{\partial Y_{i}}{\partial R}\right)^{2} \left(\frac{\partial Y_{i}}{\partial R}\right)^{2} + (D_{2}D_{6} + D_{3}D_{5}) \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial C}\right)^{2} \left(\frac{\partial Y_{i}}{\partial R}\right)^{2} \left(\frac{\partial Y_{i}}{\partial R$$

$$(M) = \frac{1}{100} (ada)^{\frac{1}{2}} \frac{1}{100} \frac{1}$$

$$S_{BN_{P=0}}^{2} = \frac{L^{2}}{D_{o}^{2}} + (D_{1}D_{8} + D_{2}D_{7}) \sum_{r=1}^{r} W_{P_{r}(obs)} \left(\frac{\partial Y_{i}}{\partial B}\right)^{\circ} \left(\frac{\partial Y_{i}}{\partial C}\right)^{\circ} \left(\frac{\partial$$

$$S_{\text{CN}_{\text{P}=0}}^{2} = \frac{L^{2}}{D_{0}^{2}} + D_{5}D_{8} \sum_{r=1}^{r} W_{\text{P}_{\text{r}}(\text{obs})} \left[\frac{\partial Y_{\underline{i}}}{\partial C} \right]^{0} + D_{5}D_{8} \sum_{r=1}^{r} W_{\text{P}_{\text{r}}(\text{obs})} \left[\frac{\partial Y_{\underline{i}}}{\partial C} \right]^{0} \right]^{2}$$

$$+ D_{6}D_{9} \sum_{r=1}^{r} W_{\text{P}_{\text{r}}(\text{obs})} \left[\frac{\partial Y_{\underline{i}}}{\partial N_{\text{P}=0}} \right]^{0} \right]^{2}$$

$$+ (D_{4}D_{8} + D_{5}D_{7}) \sum_{r=1}^{r} W_{\text{P}_{\text{r}}(\text{obs})} \left(\frac{\partial Y_{\underline{i}}}{\partial B} \right)^{0} \left(\frac{\partial Y_{\underline{i}}}{\partial C} \right)^{0}$$

$$+ (D_{4}D_{9} + D_{6}D_{7}) \sum_{r=1}^{r} W_{\text{P}_{\text{r}}(\text{obs})} \left(\frac{\partial Y_{\underline{i}}}{\partial B} \right)^{0} \left(\frac{\partial Y_{\underline{i}}}{\partial N_{\text{P}=0}} \right)^{0}$$

$$+ (D_{5}D_{9} + D_{6}D_{8}) \sum_{r=1}^{r} W_{\text{P}_{\text{r}}(\text{obs})} \left(\frac{\partial Y_{\underline{i}}}{\partial C} \right)^{0} \left(\frac{\partial Y_{\underline{i}}}{\partial N_{\text{P}=0}} \right)^{0}$$

We have indicated that the solutions to equations (14), (15), and (16) are solved by an iterative procedure. When we have the correct solutions, our linearized normal equations will be satisfied exactly. Therefore, once the best values for the constants have been determined, the remaining questions to be answered are:

(1) what is the variance of the calculated P's and any other calculated P that reduces F to zero?; and (2) what is the variance of the compressibility factor?

EVALUATION OF THE VARIANCE OF THE P_{r(calc)}'S AND ANY OTHER CALCULATED P THAT REDUCES F TO ZERO

The variance of a calculated P_r which reduces F to zero for a given observed r value is obtained in the following way: $P_r(calc)$ is a function of the observed r's and, through the constants evaluated, is a function of all of the $P_r(obs)$'s. When we apply the law for the "Propagation of Errors" to obtain an expression for determining the variance of $P_r(calc)$, we see from equation (49) that this involves evaluation of the quantity

$$s_{P_{r(calc)}}^{2} = \sum_{r=1}^{r} \left(\frac{\partial P_{r(calc)}}{\partial P_{r(obs)}}\right)^{2} s_{P_{r(obs)}}^{2}$$
(57)

In order to evaluate the variance of the calculated P_r 's, we need an expression for $[\partial P_r(calc)]^{-1}$ This quantity can be determined from equation (9)

$$F = F(r, P_{r(calc)}, N_{P=0}, B, C) \equiv 0$$
 (9)

Suppose we differentiate equation (9) with regard to $P_{r(obs)}$, holding r constant. This gives us

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and (16) are noticed by an iterative processes when we have an account contents and the content contents and an account to the contents have been descripted, the contents of the best values for the contents of the contents

DESCRIPTION OF THE VALUE OF THE PARTY PROPERTY.

The variance is a calculated P, which reduces T to note to T and a street of the a formation of the continue way: Friends is a function of the character of the and character way the continue continue continue to a function of all of the Friends as when we apply the determining the variance of Friends to obtain an expression for their this involves evaluation of the quantity. We see from equation (49)

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Suppose we dilicronstate squaries (2) with regard to Printed and Ambridge of Colors of the Colors of

$$\frac{\left(\frac{\partial F}{\partial P_{r(calc)}}\right)_{B,C,r,N_{p=0}}}{\left(\frac{\partial P_{r(calc)}}{\partial P_{r(obs)}}\right) + \left(\frac{\partial F}{\partial B}\right)_{C,r,N_{p=0},P_{r(calc)}}} + \left(\frac{\partial B}{\partial P_{r(obs)}}\right) + \left(\frac{\partial B}{\partial P_{r(obs)}}\right) + \left(\frac{\partial F}{\partial N_{p=0}}\right)_{B,C,r,P_{r(calc)}}} = 0 \quad (58)$$

Solving equation (58) for $[\partial P_{r(calc)}/\partial P_{r(obs)}]$, we get

$$\frac{\left(\frac{\partial F}{\partial B}\right)_{C,r,N_{p=0},P_{r(calc)}} \left(\frac{\partial B}{\partial P_{r(obs)}}\right)}{+\left(\frac{\partial F}{\partial C}\right)_{B,r,N_{p=0},P_{r(calc)}} \left(\frac{\partial C}{\partial P_{r(obs)}}\right)} + \frac{\left(\frac{\partial F}{\partial N_{p=0}}\right)_{B,C,r,P_{r(calc)}} \left(\frac{\partial N_{p=0}}{\partial P_{r(obs)}}\right)}{\left(\frac{\partial F}{\partial P_{r(obs)}}\right)_{B,C,r,N_{p=0}}}$$

$$\frac{\left(\frac{\partial F}{\partial P_{r(calc)}}\right)_{B,C,r,N_{p=0}}}{\left(\frac{\partial F}{\partial P_{r(calc)}}\right)_{B,C,r,N_{p=0}}}$$
(59)

Multiplying equation (59) by $S_{\stackrel{}{r}(obs)}$, squaring the product, and then summing over all of the observed $P_{\stackrel{}{r}}$'s, we get

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full spiring over all of the observed P's, we get

$$S_{B}^{2} \left(\frac{\partial F}{\partial B}\right)_{r,C,N_{P=0},P_{r}(calc)}^{2} + S_{C}^{2} \left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r}(calc)}^{2}$$

$$+ S_{N_{P=0}}^{2} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r}(calc)}^{2}$$

$$+ 2S_{BC}^{2} \left[\left(\frac{\partial F}{\partial B}\right)_{r,C,N_{P=0},P_{r}(calc)}^{2} \left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r}(calc)}^{2} \right]$$

$$+ 2S_{BN_{P=0}}^{2} \left[\left(\frac{\partial F}{\partial B}\right)_{r,C,N_{P=0},P_{r}(calc)}^{2} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r}(calc)}^{2} \right]$$

$$+ 2S_{CN_{P=0}}^{2} \left[\left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r}(calc)}^{2} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r}(calc)}^{2} \right]$$

$$+ 2S_{CN_{P=0}}^{2} \left[\left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r}(calc)}^{2} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r}(calc)}^{2} \right]$$

$$+ 2S_{CN_{P=0}}^{2} \left[\left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r}(calc)}^{2} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r}(calc)}^{2} \right]$$

$$+ 2S_{CN_{P=0}}^{2} \left[\left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r}(calc)}^{2} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r}(calc)}^{2} \right]$$

$$+ 2S_{CN_{P=0}}^{2} \left[\left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r}(calc)}^{2} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r}(calc)}^{2} \right]$$

$$+ 2S_{CN_{P=0}}^{2} \left[\left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r}(calc)}^{2} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r}(calc)}^{2} \right]$$

$$+ 2S_{CN_{P=0}}^{2} \left[\left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r}(calc)}^{2} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r}(calc)}^{2} \right]$$

$$+ 2S_{CN_{P=0}}^{2} \left[\left(\frac{\partial F}{\partial C}\right)_{r,B,N_{P=0},P_{r}(calc)}^{2} \left(\frac{\partial F}{\partial N_{P=0}}\right)_{r,B,C,P_{r}(calc)}^{2} \right]$$

from which we can evaluate the (n-1) values of S_{P}^{2} corresponding to the observed r's.

Now we ask ourselves the question: how do we calculate the variance of any other calculated P that exactly satisfies equation (9)? In order to answer this question, we must find a value of r, say r_p , and we must find a value of f, say f_p , which exactly satisfy the equation

$$Z_{P_{\text{(calc)}}} = (Z_{o}/P_{o}) f_{P} N_{P=0}^{r_{P}} P_{\text{(calc)}}$$
 (61)

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where

$$Z_{P(calc)} = 1 + BP(calc) + CP^{2}(calc)$$
 (62)

Now suppose we have previously evaluated B, C, and $N_{P=0}$ by some means or other. We now proceed to evaluate an f, say f_p , which will exactly satisfy equation (61). We do this as follows: from equation (38), f is given to be

$$f_{\text{(calc)}} = \frac{(1 + \alpha P_{1}(\text{calc}))(1 + \alpha P_{2}(\text{calc})) \dots (1 + \alpha P_{r}(\text{calc}))}{(1 + \beta P_{0})(1 + \beta P_{1}(\text{calc})) \dots (1 + \beta P_{r-1}(\text{calc}))}$$
(38)

Briggs (4) has indicated that the constants α and β which are ..." dependent on the dimensions of V_1^o and V_2^o , Poisson's ratio for V_1 and V_2 , and Young's modulus for V_1 and V_2 " ... have the following values at 30° C:

$$\alpha = 1.1463 \times 10^{-7}, \text{ psi}^{-1}$$

$$\beta = 1.1457 \times 10^{-7}, \text{ psi}^{-1}$$
 at 30° C

for the Burnett apparatus located in room 211A of the Helium Research Center. Now assuming $\alpha = \beta$, then f of equation (38) can be written as

$$f_{(calc)} = \frac{(1 + \alpha P_{r(calc)})}{(1 + \alpha P_{o})}$$
 (63)

Now, mapped as lave previously evaluated B. C. and N_{pers} by since makes or other. We now proceed to char as follows:

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deposited on the disensions of V, and V, Poisson's ratio for V, and T, and T, and Tollowing and T, and Tollowing and T, and Tollowing values are 50° C;

for the Surnett application I waited in room 2016 of the Helling Research Contest West associates of S. Chen E of equation (38) was not seen as

or,

$$f_{(calc)} = 1 + \alpha (P_{r(calc)} - P_o)[1 - \alpha P_o + \alpha^2 P_o^2 - ...]$$

For $P_0 = 1 \times 10^4$ psi, $\alpha P_0 \approx 1.1 \times 10^{-3}$. Therefore,

$$f_{(calc)} \approx 1 + \alpha (P_{r(calc)} - P_{o}) (1 - \alpha P_{o})$$
 (64)

which is accurate to about 1 part in 10^9 . Representing the correction to $N_{P=0}$ as a function of $P_{(calc)}$ by f_P , then the value of f_P which satisfies equation (61) to about 1 part in 10^9 is

$$f_{P} = 1 + \alpha (P_{(calc)} - P_{o}) (1 - \alpha P_{o})$$
 (65)

Now r_p , the expansion number corresponding to $p_{(calc)}$ and $p_{(calc)}$ and $p_{(calc)}$ can be determined from the equation

$$r_{P} = \frac{\ln Z_{P(calc)} - \ln(Z_{o}/P_{o}) - \ln P_{(calc)} - \ln f_{P}}{\ln N_{P=0}}$$
 (66)

Equation (66) results from our taking natural logarithms of equation (61) and solving for r_p . Now that we have determined values of r and f for $P_{\text{(calc)}}$, we can proceed to evaluate $S_{P_{\text{(calc)}}}^2$, the variance of a calculated P that exactly satisfies equation (9).

To evaluate the variance of $P_{(calc)}$, we employ equation (60) where the terms involving derivatives of F are to be evaluated for $P_{(calc)}$, r_P , f_P . The expression for $[\partial F/\partial P_{(calc)}]_B$, C, r_P , $N_{P=0}$ is

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(40) (3n - 1)(3 - (alma) - 2)(1 - nFp) (alma) 5

while is decrease to about 1 years in 10°. Expresseding the correction to H₁₋₁₀ as a function of Y_(0.0.1.6.) by f_p, thun the value of f_p which materials equation (bit to should I part in 10° is

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Equation (60) results from our taking matter logarithms of equation (61) and notwing for t_p. Now that we have determined yellows of r and 1 for t₍₄₁₁₆₎, we can proceed to evaluate S_p (calc) (calc)

"In equipment the variance of P(cate) we employ equation (60) where the terms involved for P(cate) " are to be evaluated for P(cate) " are to be evaluated for

given as

$$\left(\frac{\partial F}{\partial P_{\text{(calc)}}}\right)_{B,C,r_{P},N_{P=0}} = B + 2CP_{\text{(calc)}} - \left(\frac{Z_{o}}{P_{o}}\right)_{N_{P=0}}^{(r_{P})} \frac{2f_{P}P_{\text{(calc)}} - f_{P}P_{o} - P_{\text{(calc)}}}{P_{\text{(calc)}} - P_{o}}$$
(67)

EXPRESSION FOR EVALUATING THE VARIANCE OF THE COMPRESSIBILITY FACTOR

To evaluate the variance of Z_{P} , where Z_{P} is given by (calc)

$$Z_{P_{\text{(calc)}}} = 1 + BP_{\text{(calc)}} + CP_{\text{(calc)}}^{2}$$
 (62)

and $P_{(calc)}$ is any calculated P which exactly satisfies equation (9), we apply the law for the "Propagation of Errors." This law says that since Z is a function of r_p and, through the constants evaluated, is a function of all of the original observed P_r 's, then the variance of $Z_{p_{(calc)}}$ is given as

$$s_{Z_{P(calc)}}^{2} = \sum_{r=1}^{r} \left(\frac{\partial Z_{P(calc)}}{\partial P_{r(obs)}}\right)^{2} s_{P(cbs)}^{2}$$
(68)

where $S_{Z_{P(calc)}}^{2}$ is the variance of $Z_{P(calc)}$ and $S_{P(cobs)}^{2}$ is the variance of $P_{r(obs)}$.

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DIFFERENCE OF THE VARIANCE OF THE ORDERSESSIATION OF THE ORDERSESSIATION OF THE ORDERSESSIATION OF THE ORDERSESSIATION OF THE EDWIN OF THE ORDERSESSIATION OF THE ORDERSESSIATION OF THE ORDERSESSIATION OF THE ORDERSESSIA

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and F (costs) to one calculated F which exactly satisfies equation (0), we apply the Low for the "Ecopagation of Errors." This Law mays that since since I is a formation of r_p and, through the constants evaluated. It a function of all of the original observed P_p's, could be verticed of \$2.50.

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Now in order to evaluate equation (68), we must evaluate $[\partial Z_{P_{(calc)}}^{/\partial P_{r(obs)}}]$ for each $P_{r(obs)}$, multiply this quantity by $S_{P_{r(obs)}}$, square the product, and then sum the product over $P_{r(obs)}$ all of the observed P_{r} 's. When we do this, we get

$$S_{Z_{P(calc)}}^{2} = \begin{cases} B^{2} S_{P(calc)}^{2} + 4C^{2}P_{(calc)}^{2} S_{P(calc)}^{2} + P_{(calc)}^{2} S_{B}^{2} \\ + P_{(calc)}^{4} S_{C}^{2} + 4BCP_{(calc)} S_{P(calc)}^{2} + 2BP_{(calc)} S_{B,P(calc)}^{2} \\ + 2BP_{(calc)}^{2} S_{C,P(calc)}^{2} + 4CP_{(calc)}^{2} S_{B,P(calc)}^{2} \\ + 4CP_{(calc)}^{3} S_{C,P(calc)}^{2} + 2P_{(calc)}^{2} S_{BC}^{2} \end{cases}$$

$$(69)$$

The terms $S_{B,P}^{2}$ and $S_{C,P}^{2}$, which appear in equation (69), are defined as

$$s_{B,P_{\text{(calc)}}}^{2} = \sum_{r=1}^{r} \left(\frac{\partial P_{\text{(calc)}}}{\partial P_{\text{r(obs)}}} \right) \left(\frac{\partial B}{\partial P_{\text{r(obs)}}} \right) s_{P_{\text{r(obs)}}}^{2}$$
(70)

$$s_{C,P_{\text{(calc)}}}^{2} = \sum_{r=1}^{r} \left(\frac{\partial P_{\text{(calc)}}}{\partial P_{\text{r(obs)}}} \right) \left(\frac{\partial C}{\partial P_{\text{r(obs)}}} \right) s_{P_{\text{r(obs)}}}^{2}$$
(71)

and these quantities are to be evaluated from equations (70) and (71), which we proceed to do.

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and these quantities are to be confused from equations (70) and

 $[\partial P_{(calc)}/\partial P_{r(obs)}]$ can be determined from equation (59), where the terms involving derivatives of F are to be evaluated for $P_{(calc)}$, r_P , f_P , and $[\partial F/\partial P_{(calc)}]$ is defined by equation (67). Multiplying equation (59) by $(\partial B/\partial P_{r(obs)})S_{P_{r(obs)}}^2$ and summing the product over all of the observed P_r 's, we get

$$\frac{s_{B}^{2}(\frac{\partial F}{\partial B})}{r_{P},C,N_{P=0},P_{(calc)}} + s_{BC}^{2}(\frac{\partial F}{\partial C}) + s_{BC}^{2}(\frac{\partial F}{\partial C}) + s_{BN}^{2}(\frac{\partial F}{\partial C}) + s_{BN}^{2}($$

If we multiply equation (59) by $(\partial C/\partial P_{r(obs)})^{S_{p_{r(obs)}}^{2}}$ and then sum the product over all of the observed P_{r} 's, we get

$$\frac{s_{BC}^{2}\left(\frac{\partial F}{\partial B}\right)_{r_{p},C,N_{p=0},P_{(calc)}}}{+s_{C}^{2}\left(\frac{\partial F}{\partial C}\right)_{r_{p},B,N_{p=0},P_{(calc)}}} + s_{C}^{2}\left(\frac{\partial F}{\partial C}\right)_{r_{p},B,N_{p=0},P_{(calc)}} + s_{C}^{2}\left(\frac{\partial F}{\partial N_{p=0}}\right)_{r_{p},B,C,P_{(calc)}} + s_{CN_{p=0}}^{2}\left(\frac{\partial F}{\partial N_{p=0}}\right)_{r_{p},B,C,P_{(calc)}} + s_{CN_{p=0}}^{2}\left(\frac{\partial F}{\partial N_{p=0}}\right)_{r_{p},B,C,N_{p=0}} + s_{CN_{p=0}}$$

[6F (calc) 'Fr (calc)' and [6F/6F (calc)') is defined by equation (5F).

for F (calc)' Fr fr, and [6F/6F (calc)') is defined by equation (6F). Multiplying equation (5F) by (0B/6F (colc)) Fr (colc)' recommended to the product over all of the observed F's, we get

If we mulciply equation (50) by (8G/8F (obs)) E (obs) and then sum the product over all of the observed F's, we get

$$S_{BC}^{L}\left(\frac{\partial p}{\partial R}\right)^{P}_{Conlc}\left(\frac{\partial p}{$$

Therefore, the use of equations (72) and (73) in equation (69) will enable the variance of a calculated Z_p , $S_{Z_p}^2$, to be determined.

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October 1904, pp. 500-573.

Therefore, the use of equations (72) and (74) to equation (89) will enable the variance of a calculated 2, 5 (cels)

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